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A Note on Energy of Order Prime Graph of a Finite Group

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Abstract—The order prime graph $OP(\Gamma)$ of a fnite group Γ is defined as a graph with the vertex set $V(OP(\Gamma)) = \Gamma$ and two vertices a and b are adjacent in $OP(\Gamma)$ if and only if $(o(a), o(b)) = 1$. The concept of order prime graph was introduced by M. Sattanathan and R. Kala (2009). The energy of a graph is the sum of the absolute values of the eigenvalues of the adjacency matrix of the graph. The concept of energy of a graph was introduced by I.Gutman (1978). In this paper, we discuss some results on eigenvalues and energy of order prime graphs of finite groups.

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1 Introduction

For standard terminology and notion in group theory and graph theory, we refer the reader to the text-books of Herstein [7] and Harary [6] respectively. The non-standard will be given in this paper as and when required.

Throughout this paper, Γ denotes a fnite group. The order of an element a in a group Γ is denoted by $o(a)$ and order of Γ is denoted by $o(\Gamma)$. The greatest common divisor (gcd) of two numbers x and y is denoted by (x, y). The set of all positive integers is denoted by Z^+.

In [12], M. Sattanathan and R. Kala defined the order prime graphs of finite groups and studied some properties of order prime graphs. The order prime graph $OP(\Gamma)$ of a finite group Γ of order n is defined as a graph with the vertex set $V(OP(\Gamma)) = \Gamma$ and two vertices a and b are adjacent in $OP(\Gamma)$ if and only if $(o(a), o(b)) = 1$. Further, Ma et al [10] have studied order prime graphs of finite groups but by calling them as coprime graphs.

Both groups and graphs play vital roles in many theories related to chemistry. This motivated us to study eigenvalues and energy of order prime graphs. In this paper, we discuss some results on eigenvalues and energy of order prime graphs of finite groups.

We recall the following basic definitions and results:
Let G be a graph with n vertices $v_1, v_2, \dots, v_n$. The adjacency matrix $A = A(G)$ is a square matrix of order n whose (i, j)-entry is defined as

$$A_{ij} = \begin{cases} 1, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent;} \\ 0, & \text{otherwise.} \end{cases}$$

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The eigenvalues of $A(G)$ are said to be the eigenvalues of the graph G. We denote largest and smallest eigenvalues of a graph G by $\lambda_{\max}$ and $\lambda_{\min}$ respectively.

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of a graph G. The energy of G [4] is defined as,

$$E(G) = \sum_{i=1}^{n} |\lambda_i| \tag{1}$$

For a graph G with n vertices and m edges, McClelland inequality [11] is

$$E(G) \leq \sqrt{2mn} \tag{2}$$

The following inequalities (3) and (4) are given by Koolen and Moulton [8, 9]:

If G is a graph with n vertices,

$$E(G) \leq \frac{n(\sqrt{n}+1)}{2} \tag{3}$$

and if G is a bipartite graph with n vertices, where $n > 2$,

$$E(G) \leq \frac{n(\sqrt{n}+\sqrt{2})}{\sqrt{8}} \tag{4}$$

2 Some Results

Theorem 1. *If Γ is a group of order n, then*

$$\max\left\{\frac{2n-2}{n}, \sqrt{n-1}\right\} \leq \lambda_{\max}\big(OP(\Gamma)\big) \leq n-1 \tag{5}$$

In particular, if $n \geq 3$, $\sqrt{n-1} \leq \lambda_{\max}\big(OP(\Gamma)\big) \leq n-1$.

Proof. We have, if H is a subgraph of a graph G,

$$\lambda_{\max}(G) \geq \lambda_{\max}(H) \tag{6}$$

Also, for any graph G, we have

$$\max\{\bar{d}, \sqrt{d_{\max}}\} \leq \lambda_{\max} \leq d_{\max} \tag{7}$$

where $\bar{d}$ is the average vertex degree and $d_{\max}$ is the maximum vertex degree of G.

Since $d_{\max}(OP(\Gamma)) = n-1$, $K_{1,n-1}$ is a subgraph of $OP(\Gamma)$. Hence from the inequalities (6) and (7), it follows that,

$$\max\left\{\frac{2n-2}{n}, \sqrt{n-1}\right\} \le \lambda_{\max}(K_{1,n-1}) \le \lambda_{\max}(OP(\Gamma)) \le n-1.$$

Clearly, for $n \ge 3$,

$$\max\left\{\frac{2n-2}{n}, \sqrt{n-1}\right\} = \sqrt{n-1}$$

and from (5), we have, $\sqrt{n-1} \le \lambda_{\max}(OP(\Gamma)) \le n-1$.

Theorem 2. *Let Γ be a group of finite order. Then*

(i) $o(\Gamma) = p^\alpha$ where p is a prime and $\alpha \in Z^+$ if and only if for each eigenvalue λ of $OP(\Gamma)$, $-\lambda$ is an eigenvalue with the same multiplicity.

(ii) $o(\Gamma) = p^\alpha$ where p is a prime and $\alpha \in Z^+$ if and only if $\lambda_{min}(OP(\Gamma)) = -\lambda_{max}(OP(\Gamma))$.

Proof. We have, $o(\Gamma) = n = p^\alpha$, p is a prime and $\alpha \in Z^+$ if and only if $OP(\Gamma) \cong K_{1,n-1}$, a bipartite graph [12, Theorem 2.7]. Hence by [2, Proposition 3.4.1, p.38], the proof of (i) and (ii) follows.

Theorem 3. *Let Γ be a finite group of order $n \ge 3$, then $OP(\Gamma)$ has atleast three distinct eignvalues.*

Proof. A connected graph with diameter d, has at least d+1 distinct eigenvalues [2, Proposition 1.3.3, p.5]. Since $o(\Gamma) = n \ge 3$, by [10, Propositions 2.1, 2.3], it follows that the diameter of $OP(\Gamma)$ is 2. Hence $OP(\Gamma)$ has atleast three distinct eigenvaues.

Definition 4. *Let Γ be a group of finite order. The order prime energy of the group Γ, denoted by $OPE(\Gamma)$, is defined as the energy of the order prime graph $OP(\Gamma)$. That is, $OPE(\Gamma) = E(OP(\Gamma))$.*

Theorem 5. *Let Γ be a finite group.*

(i) If $o(\Gamma) = 2$, then $OP(\Gamma) \cong K_2$ and $OPE(\Gamma) = 2$.

(ii) If $o(\Gamma) = n$, then

$$OPE(\Gamma) \le \frac{n(\sqrt{n}+1)}{2} \quad (8)$$

(iii) If $o(\Gamma) = n = p^\alpha$ where p is a prime and $\alpha \in Z^+$, then

$$OPE(\Gamma) \le \frac{n(\sqrt{n}+\sqrt{2})}{\sqrt{8}} \quad (9)$$

(iv) If $o(\Gamma) = n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ where p_i's are primes and $n_i \in Z^+, \forall i$, then

$$OPE(\Gamma) \le \sqrt{n\left(n^2 - n + \sum_{i=1}^{k} x_i - \sum_{i=1}^{k} x_i^2\right)} \quad (10)$$

where x_i is the number of elements in Γ of order p_i^j, $1 \le j \le n_i$, $1 \le i \le k$.

Proof.

(i) Follows by direct computation.

(ii) Follows from the Koolen and Moulton inequality (3).

(iii) If Γ is a group of order $n = p^\alpha$ where p is a prime and $\alpha \in Z^+$, then $OP(\Gamma) \cong K_{1,n-1}$, a bipartite graph [12, Theorem 2.7]. And the inequality (9) follows from the Koolen and Moulton inequality (4).

(iv) Let m be the number of edges in $OP(\Gamma)$. Then by [12, Theorem 2.17],

$$m \le \frac{1}{2}\left(n^2 - n + \sum_{i=1}^{k} x_i - \sum_{i=1}^{k} x_i^2\right)$$

Using this inequality in McClelland inequality (1), we obtain the inequality (10).

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